Plurisubharmonic functions in calibrated geometry and q-convexity

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Abstract

Let (M, ω) be a Kähler manifold. An integrable function φ on M is called ω^q-plurisubharmonic if the current $dd^c\varphi \wedge \omega^{q-1}$ is positive. We prove that φ is ω^q-plurisubharmonic if and only if φ is subharmonic on all q-dimensional complex subvarieties. We prove that a ω^q-plurisubharmonic function is q-convex, and admits a local approximation by smooth, ω^q-plurisubharmonic functions. For any closed subvariety $Z \subset M$, $\dim_{\mathbb{C}} Z \leqslant q - 1$, there exists a strictly ω^q-plurisubharmonic function in a neighbourhood of Z (this result is known for q-convex functions). This theorem is used to give a new proof of Sibony's lemma on integrability of positive closed (p, p)-forms which are integrable outside of a complex subvariety of codimension $\geqslant p + 1$.

Contents

1 Introduction

1.1 Positive forms and ω^q-plurisubharmonic functions

The notion of ω^q-plurisubharmonic function has many facets, and these functions can be defined in many different ways. The main reason this notion is considered comes from the theory of plurisubharmonic functions on calibrated manifolds (see Subsection 2.1). However, ω^q-plurisubharmonic functions are very useful even outside of the theory of calibrations.

Let (M, ω) be a Kähler manifold, and $\varphi : M \longrightarrow \mathbb{R}$ a smooth function. Consider the form $dd^c\varphi \in \Lambda^{1,1}(M)$,

$$dd^c\varphi = 2\sqrt{-1}\,\partial\overline{\partial}\varphi.$$

Using the standard linear-algebraic argument, in a contractible open set U we might find an orthonormal frame $\xi_1, ...\xi_n \in \Lambda^{1,0}(U)$, such that

$$dd^c\varphi = -\sqrt{-1} \sum_i \alpha_i \xi_i \wedge \overline{\xi}_i. \tag{1.1}$$

Here, α_i are real functions, and the set $\{\alpha_i\}$ is independent from the choice of a frame. A function φ is called ω^q-**plurisubharmonic** if the sum of any q eigenvalues is positive:

$$\sum_{k=1}^{q} \alpha_{i_k} \geqslant 0, \tag{1.2}$$

for any k-tuple $i_1 < i_2 < ... < i_q$. This condition implies that at most $q-1$ eigenvalues of $dd^c\varphi$ can be negative. Such functions are known as q-**convex**. Indeed, ω^q-plurisubharmonicity is in many aspects similar to q-convexity.

For $q = 1$, being ω^q-plurisubharmonic is equivalent to being plurisubharmonic.

Recall that a real (p, p)-form η on a complex manifold is called **weakly positive** if for any complex subspace $V \subset T_c M$, $\dim_{\mathbb{C}} V = p$, the restriction $\rho\big|_V$ is a non-negative volume form. Equivalently, this means that

$$(\sqrt{-1})^p \rho(x_1, \overline{x}_1, x_2, \overline{x}_2, ... x_p, \overline{x}_p) \geqslant 0,$$

for any vectors $x_1, ... x_p \in T_x^{1,0} M$. A form is called **strongly positive** if it can be expressed as a sum

$$\eta = (-\sqrt{-1})^p \sum_{i_1,...i_p} \alpha_{i_1,...i_p} \xi_{i_1} \wedge \overline{\xi}_{i_1} \wedge ... \wedge \xi_{i_p} \wedge \overline{\xi}_{i_p},$$

running over some set of p-tuples $\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_p} \in \Lambda^{1,0}(M)$, with $\alpha_{i_1,...i_p}$ real and non-negative functions on M.

The strongly positive and the weakly positive forms form closed, convex cones in the space $\Lambda^{p,p}(M, \mathbb{R})$ of real (p, p)-forms. These two cones are dual with respect to the Poincare pairing

$$\Lambda^{p,p}(M, \mathbb{R}) \times \Lambda^{n-p,n-p}(M, \mathbb{R}) \longrightarrow \Lambda^{n,n}(M, \mathbb{R})$$

where $n = \dim_{\mathbb{C}} M$. For (1,1)-forms and $(n-1, n-1)$-forms, the strong positivity is also equivalent to weak positivity.

Throughout this paper, we are mostly interested in differential forms of type $\eta = \nu \wedge \omega^k$, where $\nu \in \Lambda^{1,1}(M)$ is a real (1,1)-form. For such forms, strong positivity is equivalent to weak positivity.

Claim 1.1: Let (M, ω) be a Kähler manifold, and $\nu \in \Lambda^{1,1}(M)$ a real (1,1)-form. Then, for any $k \geqslant 0$, $\nu \wedge \omega^k$ is weakly positive if and only if it is strongly positive.

Proof: Consider a decomposition similar to (1.1),

$$\nu = -\sqrt{-1} \sum_i \alpha_i \xi_i \wedge \overline{\xi}_i.$$

Then $\nu \wedge \omega^k$ can be written as

$$\nu \wedge \omega^k = (\sqrt{-1})^k \sum_{i_1 < i_2 < ... < i_{k+1}} \left(\sum_{j=1}^{k+1} \alpha_{i_j} \right) \xi_{i_1} \wedge \overline{\xi}_{i_1} \wedge \xi_{i_2} \wedge \overline{\xi}_{i_2} \wedge ... \wedge \xi_{i_{k+1}} \wedge \overline{\xi}_{i_{k+1}},$$

$$(1.3)$$

where the first sum is taken over all $(k+1)$-tuples.

For such a form, weak positivity is equivalent to strong positivity and equivalent to positivity of all coefficients $\sum_{j=1}^{k+1} \alpha_{i_j}$. ∎

From (1.3) it is clear that φ is ω^q-plurisubharmonic if and only if $dd^c\varphi \wedge \omega^{q-1}$ is positive (see Theorem 2.4 for a detailed argument).

In Theorem 3.10, we prove that a function φ is ω^q-plurisubharmonic if and only if φ is subharmonic on all germs of q-dimensional complex subvarieties of M.

For a domain in $\mathbb{C}^n$, the notion of ω^q-plurisubharmonicity was introduced and studied by Z. Khusanov, under the name "q-subharmonic" ([Kh1], [Kh2], [Kh3]). These functions were considered by Z. Błocki in [Bl], also for domains in $\mathbb{C}^n$, and used in connection with a generalized version of Monge-Ampere equation, called "complex Hessian equation".

1.2 Strictly positive forms

Let (M, ω) be a Kähler manifold. The sets of strongly and weakly positive (p, p)-forms form cones $\mathcal{C}_s$ and $\mathcal{C}_w$ in $\Lambda^{p,p}(M)$. We consider $\mathcal{C}_s$, $\mathcal{C}_w$ as topological spaces with open-compact topology. A form η lies in the interior part of $\mathcal{C}_s$, $\mathcal{C}_w$, if for any compact $K \subset M$ there exists $\varepsilon > 0$ such that $(\eta - \varepsilon\omega^p)\big|_K$ lies in $\mathcal{C}_s$, $\mathcal{C}_w$. Such a form is called **strictly positive** (strictly strongly positive and strictly weakly positive).

Exhausting M with compact sets, we obtain that η is strictly positive if and only if there exists a continuous positive function $h : M \longrightarrow]0, \infty[$ such that $\eta - h\omega^p$ is positive.

A function φ is called **strictly ω^q-plurisubharmonic** if $dd^c\varphi \wedge \omega^{q-1}$ is strictly positive (strong positivity and weak positivity are equivalent in this situation: see Claim 1.1). This is equivalent to the strictness of the inequality (1.2). For $q = 1$, this gives a definition of strictly plurisubharmonic functions.

1.3 Continuous ω^q-plurisubharmonic functions

For any locally integrable form η, $dd^c\eta$ is a well defined current. An integrable function f is called plurisubharmonic if $dd^c f$ is a positive current. Continuous strictly plurisubharmonic functions can be approximated by smooth strictly plurisubharmonic functions. This approximation is performed by taking coordinates and smoothing a function by a convolution with a smoothing kernel.

For ω^q-plurisubharmonic functions, $q > 0$, this argument fails. The smoothing operation described above is a convolution with a smooth kernel on $G \times M$, where G is a group of local automorphisms of M. For such a convolution to be smooth, this group should act transitively in a neghbourhood of a point. However, for $q > 0$ a notion of ω^q-plurisubharmonicity depends on the choice of a Kähler form, and for ω sufficiently general, (M, ω) has no local isometries.

A similar problem arises when one tries to approximate continuous subharmonic functions on arbitrary Riemannian manifold by smooth subharmonic functions. Recall that a L^1-integrable function f is **subharmonic** if the current Δf is positive. On a Kähler manifold (M, ω), $\dim_{\mathbb{C}} M = n$, a function is ω^n-plurisubharmonic if and only if it is subharmonic; this is clear, because $\Delta f \wedge \omega^n = dd^c f \wedge \omega^{n-1}$.

Green and Wu ([GW]) used the heat equation to prodice smooth approximations for all continuous subharmonic functions. We modify their argument, obtaining a smooth approximation for all strictly ω^q-plurisubharmonic functions (Theorem 3.4, Theorem 3.19).

1.4 Strictly ω^q-plurisubharmonic functions in a neighbourhood of a subvariety

One of utilities of q-convex functions comes from the following theorem.

Theorem 1.2: Let M be a complex manifold, and $Z \subset M$ a closed complex subvariety, $\dim_{\mathbb{C}} Z \leqslant q$. Then there exists an open neighbourhood $V \supset Z$ and a smooth strictly q-convex function $\varphi : V \longrightarrow R$.

Proof: [Ba], [D2]. ∎

However, q-convex functions are not very convenient to use. A sum, maximum, or regularized maximum of two q-convex functions is no longer q-convex. Also, continuous q-convex function do not admit smooth approximations.

On contrast, a sum, maximum, and regularized maximum of ω^q-plurisubharmonic functions is again ω^q-plurisubharmonic; smooth approximation is also possible.

Existence of strictly q-convex exhaustion functions leads to well-known restrictions on topology and geometry of a manifold M. Using the Morse theory, such a function gives a cell decomposition of this manifold. This can be used to show that M is homotopy equivalent to a CW-complex of

dimension at most $\dim_{\mathbb{C}} M + q - 1$. Even more information can be obtained using vanishing theorems obtained from L^2-estimates (see e.g. [D2], [O]).

For $q > 1$, ω^q-plurisubharmonicity is obviuosly stronger than q-convexity. However, it seems that there are no additional geometric restrictions on existence of strictly ω^q-plurisubharmonic functions. At least, a version of Theorem 4.1 is valid in this case.

Theorem 1.3: Let $Z \subset M$ be a closed complex subvariety of a Kähler manifold. Then there exists a strictly ω^q-plurisubharmonic function φ on an open neighbourhood of Z in M.

Proof: This is Proposition 4.5. ∎

When Z is compact, φ can be chosen exhausting, in appropriate neighbourhood of Z (Theorem 4.1).

1.5 Similar results in literature

After this paper was finished, Mihnea Colţoiu sent me the reference to a number of papers were similar results were discussed.

The strictly ω^q-plurisubharmonic functions were studied by H. Wu, in [W], under the name of *class* $\Psi(q)$-*functions*. Wu has shown that strictly ω^q-plurisubharmonic functions admit smooth approximations (Theorem 3.19).

In [NR], T. Napier and M. Ramachandran applied results of Wu to prove Bochner-Hartogs dichotomy for certain classes of Kähler manifolds, showing that a Kaehler manifold which admits a continuous exhausting plurisubharmonic function and has exactly one end either admits a holomorphic map to a Riemannian surface, or satisfies $H^1_c(M, \mathcal{O}) = 0$, where $H^1_c(M, \mathcal{O})$ is a group of holomorphic cohomology with compact support. Napier and Ramachandran consider ω^q-plurisubharmonic functions, under the name of *q-plurisubharmonic functions*. They state a version of Theorem 4.1 and sketch its proof, referring to Demailly and Colţoiu, who proved a similar result for q-convex functions in [D2] and [C].

2 ω^q-plurisubharmonic functions

2.1 Plurisubharmonic functions in calibrated geometry

Since their introduction by Harvey and Lawson in [HL1], calibrated geometries became a mainstay of much of modern differential geometry and some

of string physics. Through calibrations, the arguments of Kähler geometry can be applied to manifolds with special holonomy, and other interesting differential-geometric structures.

Definition 2.1: Let M be a Riemannian manifold. A **calibration** on M is a closed real k-form $\rho \in \Lambda^k(M)$ such that for any k-dimensional subspace $V \subset T_x M$, we have

$$\left| \rho|_V \right| \leqslant |\operatorname{Vol}_V|, \tag{2.1}$$

where Vol_V is a Riemannian volume form on V.

A space $V \subset T_x M$ is called **calibrated by** ρ is $\left| \rho|_V \right| = |\operatorname{Vol}_V|$. A k-dimensional submanifold $Z \subset M$ is **calibrated by** ρ if $T_z Z \subset T_z M$ is calibrated, for all $z \in Z$.

Examples of well-known calibrations include the Kähler form and its powers (normalized as $\frac{1}{q!}\omega^q$); the fundamental 3-form and 4-form on a G_2-manifold; real part of the holomorphic volume form on a Calabi-Yau manifold; and so on.

For a detailed study of calibrations on manifolds, the reader can consult the books [Ha] and [J].

The main utility of calibrations is due to the following important theorem.

Theorem 2.2: ([HL1]) Let (M, g, ρ) be a Riemannian manifold equipped with a calibration, and $Z \subset M$ an oriented submanifold. Then

$$\int_Z \operatorname{Vol}_Z \geqslant \left| \int_Z \rho \right|.$$

Moreover, the equality happens only when Z is calibrated by ρ.

Proof: Follows immediately from (2.1). ∎

This result is remarkable, because for Z compact, the integral $\int_Z \rho$ is a cohomological invariant. Therefore, compact calibrated manifolds are minimal, and, moreover, minimize the Riemannian volume in their cohomology class.

When $\rho = \frac{1}{q!}\omega^q$, where ω is a Kähler form, a manifold $Z \subset M$ is calibrated with respect to ρ if and only if Z is complex analytic.

In [HL2], Harvey and Lawson defined the notion of a plurisubharmonic function associated with a calibration. In the present paper, we are interested in parallel calibrations, that is, calibrations satisfying $\nabla \rho = 0$, where ∇ is the Levi-Civita connection. In [HL2], plurisubharmonic functions are defined with respect to an arbitrary calibration, but for parallel ones the definition is simpler. We restrict ourselves to parallel calibrations.

Consider a parallel calibration $\rho \in \Lambda^k(M)$, and let $\Lambda^1(M) \xrightarrow{\;C\;} \Lambda^{k-1}(M)$ be the map $\theta \longrightarrow \rho \lrcorner \theta^\sharp$, where $\lrcorner$ is a contraction, and $\theta^\sharp$ the dual vector field. When ρ is the Kähler form, this is a complex structure operator; when ρ is the fundamental 3-form on a G_2-manifold, C is dual to the vector product ([V2]).

Consider an operator $d_c := \{d, C\}$, where $\{\ \}$ denotes the supercommutator,

$$\{d, C\} = dC - (-1)^{\deg \rho} Cd.$$

The operator $dd_c : \; C^\infty M \longrightarrow \Lambda^k M$ (introduced independently in [V2]) is a natural analogue of the operator dd^c in complex geometry. When ρ is a Kähler form, dd_c is equal to dd^c.

Given a calibrated subspace $V \subset T_x M$, $\rho\big|_V$ is non-degenerate, and gives an orientation. A smooth function $\varphi : \; M \longrightarrow \mathbb{R}$ is called **plurisubharmonic with respect to** ρ, if $dd_c\varphi$ is positive on any calibrated subspace $V \subset T_x M$. We extend this definition to integrable functions in Subsection 3.1. When the restriction $dd_c\varphi\big|_V$ is non-degenerate on all ρ-calibrated subspaces, φ is called **strictly plurisubharmonic with respect to** ρ.

We are going to explore this notion in the case when $\rho = \frac{1}{q!}\omega^q$, where ω is a Kähler form.

2.2 ω^q-plurisubharmonic functions and q-convexity

From now on, (M, ω) is a Kähler manifold, considered as a calibrated manifold with the calibration $\rho = \frac{1}{q!}\omega^q$. A function is called ω^q-**plurisubharmonic** if it is plurisubharmonic with respect to the calibration $\frac{1}{q!}\omega^q$.

Claim 2.3: In these assumptions,

$$d_c(\varphi) = \frac{1}{(q-1)!} d^c\varphi \wedge \omega^{q-1},$$

where $d^c = IdI^{-1}$ is the usual twisted de Rham differential.

Proof: Clearly,

$$d^c\varphi = \omega \lrcorner (d\varphi)^\sharp,$$

and

$$d_c\varphi = q\frac{1}{q!}\omega^{q-1} \wedge (\omega \lrcorner (d\varphi)^\sharp).$$

From these two equations, Claim 2.3 clearly follows. ∎

From Claim 2.3, we obtain that the ω^q-plurisubharmonicity of a function φ depends on the pseudo-Hermitian $(1,1)$-form $dd^c\varphi$. In some orthonormal basis $\xi_1, ...\xi_n$ in $\Lambda_z^{1,0}(M)$, this form can be written as $dd^c\varphi = \sqrt{-1}\sum_i \alpha_i \xi_i \wedge \overline{\xi}_i$, with α_i real numbers. We call these numbers **the eigenvalues of** $dd^c\varphi$ at $z \in M$. Clearly, the set $\{\alpha_i\}$ is independent from the choice of the orthonormal basis ξ_i.

Theorem 2.4: Let (M, ω) be a Kähler manifold, and $\varphi : M \longrightarrow \mathbb{R}$ a smooth function. Then the following conditions are equivalent.

(i) φ is ω^q-plurisubharmonic

(ii) $dd^c\varphi \wedge \omega^{q-1}$ is weakly positive

(ii) $dd^c\varphi \wedge \omega^{q-1}$ is strongly positive

(iv) Suppose we order the eigenvalues of $dd^c\varphi$ as follows: $\alpha_1 \leqslant \alpha_2 \leqslant ... \leqslant \alpha_n$. Then

$$\sum_{i=1}^q \alpha_i > 0. \tag{2.2}$$

Remark 2.5: The inequality (2.2) implies that $\alpha_q, \alpha_{q+1}, ...\alpha_n$ are non-negative. Indeed, if α_q is negative, then $\alpha_1, ..., \alpha_{q-1}$ are also negative, but then $\sum_{i=1}^q \alpha_i < 0$.

Proof of Theorem 2.4: The equivalence (i) $\Leftrightarrow$ (ii) is clear from Claim 2.3. Indeed, a plane $V \subset T_xM$ is ω^q-calibrated if and only if it is complex; therefore, φ is ω^q-plurisubharmonic if and only if $dd_c\varphi = \frac{1}{(q-1)!}dd^c\varphi \wedge \omega^{q-1}$ is weakly positive. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Claim 1.1. Now, let $\xi_1, ...\xi_n$ be an orthonormal basis in $\Lambda^{1,0}(M)$, such that $dd^c\varphi = \sqrt{-1}\sum_i \alpha_i \xi_i \wedge \overline{\xi}_i$. Then

$$\frac{1}{(q-1)!}dd^c\varphi \wedge \omega^{q-1} = \sum_{i_1 < i_2 < ... < i_q} \left(\sum_{k=1}^q \alpha_{i_k}\right)(\sqrt{-1})^q \xi_{i_1} \wedge \overline{\xi}_{i_1} \wedge ... \wedge \xi_{i_q} \wedge \overline{\xi}_{i_q}. \tag{2.3}$$

Clearly, when all the coefficients $\sum_{k=1}^q \alpha_{i_k}$ are positive, the form (2.3) is also positive, being a sum of positive monomials. The converse is also true. Indeed,

$$(\sqrt{-1}\,)^q \frac{1}{(q-1)!} dd^c\varphi \wedge \omega^{q-1}(\xi_{i_1}^\sharp, \overline{\xi}_{i_1}^\sharp, \xi_{i_2}^\sharp, \overline{\xi}_{i_2}^\sharp, ..., \xi_{i_q}^\sharp, \overline{\xi}_{i_q}^\sharp) = \sum_{k=1}^q \alpha_{i_k},$$

where $\xi_{i_2}^\sharp \in TM$ is the basis dual to ξ_i. Therefore, if one of the coefficients $\sum_{k=1}^q \alpha_{i_k}$ is negative, then $\frac{1}{(q-1)!} dd^c\varphi \wedge \omega^{q-1}$ is negative on the space $V = \langle \xi_{i_1}^\sharp, \xi_{i_2}^\sharp, ..., \xi_{i_q}^\sharp \rangle$ generated by the corresponding q-tuple.

We obtained that the positivity of the form $\frac{1}{(q-1)!} dd^c\varphi \wedge \omega^{q-1}$ is equivalent to non-negativity of all the coefficients $\sum_{k=1}^q \alpha_{i_k}$. The smallest of these coefficients is $\sum_{k=1}^q \alpha_k$. Therefore, positivity of $\frac{1}{(q-1)!} dd^c\varphi \wedge \omega^{q-1}$ is equivalent to $\sum_{k=1}^q \alpha_k \geqslant 0$. We proved Theorem 2.4. ∎

Remark 2.6: In these assumptions, φ is strictly ω^q-plurisubharmonic if and only if the sum of q smallest eigenvalues $\sum_{k=1}^q \alpha_k$ is positive everywhere; this is proven by the same argument as the above theorem.

In complex geometry, the notion of q-convexity is used quite often. A function $\varphi: M \longrightarrow \mathbb{R}$ on a complex manifold is called **strongly q-convex** (see e.g. [D2]) if $dd^c\varphi$ has at most $(q-1)$ negative or zero eigenvalues. Even if the eigenvalues of $dd^c\varphi$ are defined in terms of a Kähler (or a Hermitian) form, their sign is independent from this choice. Therefore, the q-convexity of a function depends only on the complex structure on M.

From Remark 2.6, it is clear that strictly ω^q-plurisubharmonic functions are strongly q-convex. The converse is not true: q-convexity does not necessarily imply the inequality $\sum_{k=1}^q \alpha_k > 0$ for the sum of q smallest eigenvalues of $dd^c\varphi$.

However, unlike plurisubharmonic functions, q-convex functions are not very convenient to use. A sum of two q-convex functions is no longer q-convex. Moreover, continuous q-convex functions (defined as a function which can be locally expressed as a maximum of a funite number of smooth q-convex functions)[1] do not necessarily have a smooth q-convex approximation ([DF], [P]). All of these problems are rectified if we consider ω^q-plurisubharmonic functions instead of q-convex: sum and maximum of ω^q-plurisubharmonic is again ω^q-plurisubharmonic, and any strict ω^q-plurisubharmonic function admits a smooth approximation (see Section 3).

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[1]Such functions are also known as q-**convex functions with corners**.

3   Properties of ω^q-plurisubharmonic functions

3.1   Continuous ω^q-plurisubharmonic functions

Just like it happens in the case of the usual plurisubharmonic functions, the definition of ω^q-plurisubharmonic functions can be generalized, to include continuous ω^q-plurisubharmonic functions.

   The following claim is trivial.

Claim 3.1: Let M be a Kähler manifold. A smooth function φ is ω^q-plurisubharmonic if and only if

$$\int_M \varphi \omega^{q-1} \wedge dd^c \alpha \geqslant 0, \tag{3.1}$$

for any strongly positive form α with compact support.

   Proof: We use

$$\int_M \varphi \omega^{q-1} \wedge dd^c \alpha = \int_M dd^c \varphi \wedge \omega^{q-1} \wedge \alpha,$$

which follows from Stokes' theorem. ∎

   Now, (3.1) can be used to define the ω^q-plurisubharmonicity for continuous functions (see also [HL3]).

Definition 3.2: Let (M, ω) be a Kähler manifold, and $\varphi : \ M \longrightarrow \mathbb{R}$ a locally integrable function. Then φ is called ω^q-**plurisubharmonic** if

$$\int_M \varphi \omega^{q-1} \wedge dd^c \alpha \geqslant 0$$

for any positive form α with compact support.

   From Claim 3.1, it is obvious that for smooth φ this definition agrees with the one we gave previously. Also, the following statement is apparent.

Claim 3.3: Let $\{\varphi_i\}$ be a sequence of ω^q-plurisubharmonic functions which locally converge to φ in L^1-topology. Then φ is also ω^q-plurisubharmonic. ∎

3.2   Local approximation for ω^q-plurisubharmonic functions

The ω^q-plurisubharmonic functions admit a smooth approximation, both locally (Theorem 3.4) and globally (Theorem 3.19). The arguments we use are taken from [GW], where the same result is proven for subharmonic functions on Riemannian manifolds.

Theorem 3.4: Let (M, ω) be an open Kähler manifold, biholomorphic to an open ball, $K \subset M$ a compact subset, and $\varphi : \ M \longrightarrow \mathbb{R}$ a continuous ω^q-plurisubharmonic function. Then there exists a sequence $\{\varphi_i\}$ of smooth functions on M, ω^q-plurisubharmonic in some neighbourhood of K, and uniformly converging to φ on K.

Proof: The proof of Theorem 3.4 uses the notion of a *heat kernel*. We recall the necessary results, which are well known (see e.g. [He]).

Theorem 3.5: Let M be a compact Riemannian manifold, $\dim_{\mathbb{R}} M = n$, and $\eta \in \Lambda^k(M)$ an L^1-integrable k-form. Then the equation

$$\frac{d\eta_t}{dt} = -\Delta\eta_t, \ \ \eta_0 = \eta,$$

has a unique solution $\eta_t \in \Lambda^k(M)$, smoothly depending on $t \in \mathbb{R}^{\geqslant 0}$. Moreover,

(i) For any $t > 0$, the form η_t is smooth, and as $t \longrightarrow 0$, η_t converges to η; uniformly if η is continuous.

(ii) The solution η_t can be obtained by an integration with a **heat kernel**, K_t, which is an L^1-integrable n-form on $M \times M$,

$$\eta_t = (\pi_2)_*(\pi_1^*\eta \wedge K_t),$$

where $\pi_1, \pi_2 : \ M \times M \longrightarrow M$ are natural projections, π_1^* the pullback map, and $(\pi_2)_*$ the fiberwise integration.

(iii) The form K_t is smooth, for all $t > 0$.

(iv) If η is smooth on a neighbourhood of a compact set $K \subset M$, then all derivatives of η_t uniformly converge to the respective derivatives of η on K.

Proof: See e.g. [BGV]. ∎

On a Kähler manifold, the heat kernel can be locally expressed as follows.

Theorem 3.6: Let M be a compact Kähler manifold, of complex dimension n, and $K_t \in \Lambda^{2n}(M)$ the heat kernel. Then

(i) K_t has Hodge type (n, n)

(ii) K_t can be decomposed as $K_t = K_t^\circ + K_t^{\circ\circ}$, with $K_t^{\circ\circ}$ a smooth form which converges to 0 uniformly as $t \longrightarrow 0$, and K_t° positive and equal 0 outside of a neighbourhood U of $\mathsf{diag} \subset M \times M$. Moreover, U can be chosen arbitrarily small.

(iii) Denote by H_t the operator which maps η to the solution η_t of the heat equation. Then H_t commutes with the operators $d, d^c, \partial, \overline{\partial}$, and with the multiplication by the Kähler form.

Proof: See [BGV], [He]. ∎

We are going to approximate a ω^q-plurisubharmonic function φ by $\varphi_t :=H_t(\varphi)$. To prove Theorem 3.5, we need to show that φ_t is also ω^q–plurisubharmonic. By Theorem 3.6 (iii), it is sufficient to show that the form η_t is positive for η strictly positive, and t sufficiently small.

Using this description of the heat kernel, the following proposition can be easily established.

Proposition 3.7: Let $K \subset M$ be a compact subset of a compact Kähler manifold, and $\varphi : M \longrightarrow \mathbb{R}$ an L^1-integrable function which is strictly ω^q-plurisubharmonic on some open neighbourhood of K. Consider the solutuion of a heat equation

$$\frac{d\varphi_t}{dt} = -\Delta\varphi_t, \quad \varphi_0 = \varphi, \tag{3.2}$$

Then, for t sufficiently small, φ_t is ω^q-plurisubharmonic on some open neighbourhood of K.

Proof: As we have already mentioned, to prove Proposition 3.7 it suffices to show that the solution η_t of heat equation is positive, for any current $\eta \in \Lambda^{q,q}(M, \mathbb{R})$ which is strictly positive in an open neighbourhood $U \supset K$,

and t sufficiently small. Replacing U by a smaller neighbourhood of K, we can always assume that there exists $\varepsilon > 0$ such that $\eta\big|_U > \varepsilon\omega^q$. Writing $K_t = K_t^\circ + K_t^{\circ\circ}$ as in Theorem 3.6, we can express η_t as

$$\eta_t = \eta_t^\circ + \eta_t^{\circ\circ},$$

where

$$\eta_t^\circ = (\pi_2)_*(\pi_1^*\eta \wedge K_t^\circ), \quad \eta_t^{\circ\circ} = (\pi_2)_*(\pi_1^*\eta \wedge K_t^{\circ\circ}).$$

Since K_t° is positive, and vanishes outside of a sufficiently small neighbourhood of diag, the form η_t° is also positive, in some smaller open neighbourhood $U' \supset K$. Moreover, $\eta_t^\circ > \varepsilon\omega^q$ in U' because $\eta > \varepsilon\omega^q$, and $H_t(\omega^q) = \omega^q$. Since $K_t^{\circ\circ}$ is smooth and uniformly converges to zero, $\eta_t^{\circ\circ}$ is smooth and uniformly converges to zero as $t \longrightarrow 0$. Then

$$\eta_t > \varepsilon\omega^q + \eta_t^{\circ\circ}$$

is positive for any t for which $|\eta_t^{\circ\circ}|_{L^\infty} < \varepsilon$. For any such t, φ_t is also ω^q-plurisubharmonic in a neighbourhood of K. $\blacksquare$

To finish the proof of the local approximation theorem (Theorem 3.4), we shall need the following lemma.

Lemma 3.8: Let M be an open ball in $\mathbb{C}^n$, centered in 0 and equipped with a Kähler form $\omega = dd^c\varphi$, with φ bounded on M. Fix a smaller ball $B \subsetneq M$, also centered in 0. Then M admits a holomorphic embedding to a compact Kähler manifold $(\widetilde{M}, \widetilde{\omega})$, such that $\widetilde{\omega}\big|_B = \omega\big|_B$.

Proof: In [D1], the notion of a *regularized maximum* of two functions was defined. Choose $\varepsilon > 0$, and let $\max_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth, convex function which satisfies $\max_\varepsilon(x,y) = \max(x,y)$ whenever $|x - y| > \varepsilon$. Then $\max_\varepsilon$ is called **a regularized maximum**. It is easy to show ([D1]) that a regularized maximum of two strictly plurisubharmonic functions is again strictly plurisubharmonic.

Replacing the potential φ by $\varphi + \inf_M \varphi$, we can make sure that $0 \leqslant \varphi \leqslant C$, for some $C \in \mathbb{R}$. Rescaling, we may assume that the radius of the ball B is 1, and radius of M is $R > 1$. Consider the function

$$\widetilde{\varphi} := \max_\varepsilon(\varphi, V),$$

on M, where $V(x) = A|x|^2 - A - \varepsilon$, where A is a positive number, chosen in such a way $AR^2 - A - \varepsilon > C + \varepsilon$. Then $V(x) < 0$ on B, and $V(x) > \varphi + \varepsilon$

around the boundary of M. The Kähler metric $\widetilde{\omega} = dd^c\widetilde{\varphi}$ is flat around the boundary of M, and equal φ on B. Gluing $(M, \widetilde{\omega})$ to a round hole in a complex torus $\widetilde{M}$ with flat Kähler metric, we obtain an isometric embedding $(M, \widetilde{\omega}) \hookrightarrow (\widetilde{M}, \widetilde{\omega})$, with $\widetilde{\omega}\Big|_B = \omega\Big|_B$. This proves Lemma 3.8. $\blacksquare$

Now we can finish the proof of the local approximation theorem (Theorem 3.4). First, the function φ can be approximated by a sequence $\varphi + \varepsilon|z|^2$ of strictly ω^q-plurisubharmonic functions. Therefore, it suffices to prove Theorem 3.4 assuming that φ is strictly ω^q-plurisubharmonic. Using Lemma 3.8, we can embed M into a compact torus $\widetilde{M}$, in such a way that the Kähler form $\widetilde{\omega}$ of $\widetilde{M}$ is equal to ω on some neighbourhood of K. We extend φ to a continuous function $\widetilde{\varphi}$ on $\widetilde{M}$, and use the heat equation to approximate $\widetilde{\varphi}$ by the solutions $\widetilde{\varphi}_t$ of the heat equation. By Proposition 3.7, for t sufficiently small, $\widetilde{\varphi}_t$ is also ω^q-plurisubharmonic on some neghbourhood of K. This proves the Local Approximation Theorem (Theorem 3.4). $\blacksquare$

3.3 Functions which are subharmonic on q-dimensional complex subvarieties

Let M be an n-dimensional Riemannian manifold. A smooth subharmonic function $\varphi : M \longrightarrow \mathbb{R}$ is a function which satisfies $\Delta\varphi \geqslant 0$. This is equivalent to $\int_M \Delta\varphi\alpha \geqslant 0$, for any non-negative positive volume form α with compact support. However, by Stokes' theorem

$$\int_M \Delta\varphi \cdot \alpha = \int_M *d*d\varphi \cdot \alpha = \int_M d*d\varphi \cdot *\alpha =$$
$$= -(-1)^{n-1}\int_M *d\varphi \wedge d*\alpha = -\int_M d\varphi \wedge *d*\alpha =$$
$$= \int_M \varphi \cdot d*d*\alpha = \int_M \varphi \cdot \Delta\alpha.$$

Repeating the construction of Subsection 3.1, we generalize the notion of subharmonicity.

Definition 3.9: A locally L^1-integrable function $\varphi : M \longrightarrow \mathbb{R}$ is called **subharmonic** if and only if $\int \varphi\Delta\alpha \geqslant 0$, for any smooth volume form α with compact support.

This definition is well known, see e.g. [GW].

Theorem 3.10: Let (M, ω) be a Kähler manifold, and $\varphi : M \longrightarrow \mathbb{R}$ a

continuous function. Then φ is ω^q-plurisubharmonic if and only if φ is subharmonic on all complex submanifolds of dimension q.[1]

Proof: This result is well known for the usual plurisubharmonic functions (which are ω^1-plurisubharmonic in our terminology), and its proof for general q is no different.

From [HL2], Theorem 1.4, it is known that a smooth function which is plurisubharmonic with respect to a calibration is subharmonic on all calibrated submanifolds. For a calibration $\frac{1}{q!}\omega^q$, calibrated subvarieties are precisely complex subvarieties of dimension q, hence a smooth ω^q–plurisubharmonic function is subharmonic on such subvarieties.

To prove this result for continuous plurisubarmonic functions, we use the local approximation theorem (Theorem 3.4).

To prove the converse statement, assume that φ is a continuous function on M which is subharmonic on all complex submanifolds of dimension q. The statement of Theorem 3.10 is local. Replacing M by a smaller open set if necessary, we may assume that M, as a complex manifold, is isomorphic to a product, $M = F \times B$, with $\dim F = q$. Denote by $\pi : M \longrightarrow B$ the natural projection. Let $\pi^* \mathrm{Vol}_B$ be a volume form on B lifted to M, and α a smooth function on M with compact support. The cone of strongly positive $(n - q, n - q)$-forms is locally generated by the forms of type $\alpha\pi^* \mathrm{Vol}_B$, taken for different α and π. Therefore, to prove positivity of the current $dd^c\varphi \wedge \omega^{q-1}$, it suffices to show that

$$\int \varphi \wedge \omega^{q-1} \wedge \pi^* \mathrm{Vol}\, B \wedge dd^c\alpha \geqslant 0.$$

However, this integral can be expressed using the Fubini formula, as

$$\int \varphi \wedge \omega^{q-1} \wedge \pi^* \mathrm{Vol}\, B \wedge dd^c\alpha = \int_B \mathrm{Vol}_B \int_F \varphi\omega^{q-1} \wedge dd^c\alpha. \qquad (3.3)$$

Since $\omega^{q-1} \wedge dd^c\alpha = \Delta\alpha\, \mathrm{Vol}_F$, and φ is subharmonic on the fibers of π, the second integral is always positive. Therefore, (3.3) is also positive. We proved Theorem 3.10. ∎

Remark 3.11: There is another definition of subharmonicity: a function φ on M is called subharmonic, if for any harmonic function φ_1, and any compact convex set $B \subset M$, with boundary ∂B, $\varphi\big|_{\partial B} \leqslant \varphi_1\big|_{\partial B}$ implies

---

[1] Here we consider complex subvarieties not necessarily closed in M.

$\varphi\big|_B \leqslant \varphi_1\big|_B$. This definition is equivalent to Definition 3.9, as shown in [GW], Lemma 3.1.

Corollary 3.12: A maximum of two continuous ω^q-plurisubharmonic functions is again ω^q-plurisubharmonic.

Proof: For subharmonic functions this is well known (and trivial if we use the definition mentioned in Remark 3.11). Now Corollary 3.12 is implied immediately by Theorem 3.10. ∎

In the proof of Lemma 3.8, a notion of a regularized maximum was defined, following Demailly ([D1]). It is well known that a regularized maximum of two subharmonic functions is again subharmonic. Using Theorem 3.10, the following Corollary is immediately obtained.

Corollary 3.13: A regularized maximum of two continuous ω^q–plurisubharmonic functions is again ω^q-plurisubharmonic.
∎

3.4   Global approximation for ω^q-plurisubharmonic functions

In [GW], Green and Wu have proven several powerful theorems which they used to obtain the smooth approximation for a number of important classes of functions on Riemannian manifolds, including convex, subharmonic, and Lipschitz functions. Their approach is axiomatic, based on the formal properties that are satisfied by these classes of functions.

It turns out that these formal properties are satisfied also for the sheaf of strictly ω^q-plurisubharmonic functions as well.

Let $K \subset M$ be a compact subset of a Riemannian manifold, and $f : M \longrightarrow \mathbb{R}$ a continuous function, which is smooth in a neighbourhood of K. Define

$$\mathfrak{D}(K, i, f) := \sup_K |\nabla^i f|,$$

where $\nabla : (\Lambda^1 M)^{\otimes k} \longrightarrow (\Lambda^1 M)^{\otimes k+1}$ is the Levi-Civita connection. Consider the pseudonorm

$$\|f\|_K := \sup_K |f| + \sum_{i=1}^{\infty} \frac{1}{2^i} \max(1, \mathfrak{D}(K, i, f)),$$

and the corresponding pseudometric $d_k(f, g) := \|f - g\|_K$.

Definition 3.14: Let $\mathfrak{C}$ be a subsheaf of a sheaf of continuous functions. We say that $\mathfrak{C}$ **has** C^∞**-stability property** if for any $h \in \Gamma(\mathfrak{C}, U)$, and any compact set $K \subset U$, there exists $\varepsilon > 0$, such that for any function f on U, smooth in a neighbourhood of K, and satisfying $\|f\|_K < \varepsilon$, the sum $h + f$, restricted to some neighbourhood of K, lies in $\mathfrak{C}$.

Definition 3.15: Let $\mathfrak{C}$ be a subsheaf of a sheaf of continuous functions. We say that $\mathfrak{C}$ **satisfies the semilocal approximation property**, if for any

- $\varepsilon > 0$

- open subset $U \subset M$

- compact subset $K \subset U$

- compact subset $K_1 \subset K$ (possibly, empty)

- function $f \in \Gamma(\mathfrak{C}, U)$, which is smooth in a neighbourhood of K_1

there exists a smooth function $F \in \Gamma(\mathfrak{C}, U)$, such that

- $\sup_K |f - F| < \varepsilon$

- $d_{K_1}(F, f) < \varepsilon$.

Definition 3.16: Let $\mathfrak{C}$ be a subsheaf of a sheaf of continuous functions. We say that $\mathfrak{C}$ **has local approximation property** if for each point $p \in M$, there is an open neighbourhood $U \ni p$ such that the restriction $\mathfrak{C}\big|_U$ has the semilocal approximation property.

Definition 3.17: Let $\mathfrak{C}$ be a subsheaf of a sheaf of continuous functions. We say that $\mathfrak{C}$ **has the maximum closure property** if for any sections $f, g \in \Gamma(\mathfrak{C}, U)$, the maximum $\max(f, g)$ also belongs to $\mathfrak{C}$.

The following result was proven in [GW]

Theorem 3.18: (Global Approximation Theorem; see Theorem 1.1 and Theorem 4.1 of [GW]). Let $\mathfrak{C}$ be a subsheaf of a sheaf of continuous functions satisfying the C^∞-stability, maximum closure, and local approximation

property. Then, for any $f \in \Gamma(\mathfrak{C}, M)$, and any positive function g on M, there exists a smooth function $F \in \Gamma(\mathfrak{C}, M)$, such that $|F - f| < g$.

Theorem 3.19: Let M be a Kähler manifold, and f a strictly ω^q–plurisubharmonic continuous function. Then, for any positive function g on M, there exists a smooth ω^q-plurisubharmonic function $F \in \Gamma(\mathfrak{C}, M)$, such that $|F - f| < g$.

Proof: To prove the Global Approximation Theorem for ω^q–plurisubharmonic functions, we need only to check that the sheaf of such functions satisfies the properties of Green-Wu theorem (Theorem 3.18). The maximum closure property is clear from Corollary 3.12, and C^∞-stability is clear from the definition of strict ω^q-plurisubharmonicity. The local approximation property follows from the heat equation argument given in Subsection 3.2. Indeed, the solution of heat equation φ_t converges to φ uniformly as $t \longrightarrow 0$. Moreover, if φ is smooth in a neighbourhood of a compact set K_1, then all derivatives of φ_t converge to respective derivatives of φ. We proved the global approximation theorem for continuous strictly ω^q-plurisubharmonic functions. ∎

4 ω^q-plurisubharmonic functions in a neighbourhood of a complex variety of dimension $\leqslant q - 1$

4.1 The main statement

Given a compact complex subvariety $Z \subset M$, $\dim Z < q$, in some neighbourood of Z there exists a strictly q-convex, exhausting function ([Ba], [D2]). In the present paper, we prove a similar result for ω^q-convex functions. Since all ω^q-convex functions are also q-convex, in Kähler setting our theorem also implies this classical result.

Recall that a function $\varphi : \ M \longrightarrow [-\infty, c[$ is called **exhausting** if for any $c_1 < c$ the set $\varphi^{-1}([-\infty, c_1])$ is compact.

Theorem 4.1: Let M be a Kähler manifold, and $Z \subset M$ a compact complex subvariety, $\dim_{\mathbb{C}} Z \leqslant q - 1$. Then there exists an open neighbourhood $V \supset Z$ and a smooth strictly ω^q-plurisubharmonic function $\varphi : \ V \longrightarrow R$ which is exhausting.

The proof of Theorem 4.1 roughly follows the argument used by J.-P. Demailly in [D2]. In Subsection 4.2, we construct a strictly ω^q–plurisubharmonic function in a neighbourhood of V, and in Subsection 4.3, modify it to be exhausting.

Remark 4.2: Theorem 4.1 is trivial when Z is smooth, and M complete as a metric space. Indeed, for a smooth, closed submanifold $Z \subset M$, with TZ containing no Φ-calibrated subspaces, the distance function $x \longrightarrow \mathrm{dist}(x, Z)$ is Φ-plurisubharmonic in a neighbourhood of Z. This function is exhausting in an open neighbourhood

$$\{x \in M \mid \mathrm{dist}(x, Z) < \varepsilon\},$$

for any $\varepsilon > 0$, because M is complete.

4.2   Construction of ω^q-plurisubharmonic functions in a neighbourhood of a subvariety

We use the following lemma

Lemma 4.3: Let M be a compact Riemannian manifold with boundary ∂M, and $Z \subset M$ a closed submanifold with a smooth boundary $\partial Z = Z \cap \partial M$. Assume that $M_0 := M \backslash \partial M$ is complex and Kähler, and $Z_0 := Z \cap M_0$ is also complex, $\dim_{\mathbb{C}} Z_0 \leqslant q - 1$. Assume that in a neighbourhood of each point in ∂Z, M can be represented as $Z \times B$, where B is an open ball, and this splitting is compatible with the complex structure. Then there exists a smooth function $\varphi : U \longrightarrow \mathbb{R}$, defined on some neighbourhood $U \supset Z$, which is strictly ω^q-plurisubharmonic on $U_0 = U \cap M_0$. Moreover, φ can be chosen in such a way that φ and all its derivatives vanish on $\partial M \cap U$.

Proof: Notice that only the last statement of Lemma 4.3 is non-trivial; if we did not care about the boundary, we could just take $\varphi(x) = \mathrm{dist}(x, Z)$. Passing to a covering of Z by sufficiently small open sets, we find that Lemma 4.3 is reduced to the following statement, which is a special case of Lemma 4.3.

Sublemma 4.4: Let Z be a smooth Riemannian manifold with a boundary, B an open ball in $\mathbb{C}^n$, centered in 0, and $M = Z \times B$ their product, equipped with some Riemannian structure. Assume that $M_0 = (Z \backslash \partial Z) \times B \subset M$ is Kähler, and $\dim_{\mathbb{C}} Z \leqslant q - 1$. Consider a smooth function $\theta_Z : Z \longrightarrow \mathbb{R}$,

positive everywhere outside of a boundary, and satisfying

$$|d\theta_Z| < C_1\theta_Z, \quad |\partial\overline{\partial}\theta_Z| < C_2\theta_Z,$$

for some constants C_1, C_2. Let $\varphi : \ M \longrightarrow \mathbb{R}$ map (z,b) to $\theta_Z(z)|b|^2$. Then $\left.\varphi\right|_{(Z\backslash\partial Z)\times B_1}$ is strictly ω^q-plurisubharmonic, for some open ball $B_1 \subset B$, also centered in 0, with radius depending on the constants C_1, C_2.

Proof: Let ω_B denote the flat Kähler form $\omega_B := dd^c|b|^2$ on B lifted to M, and $\theta : \ M \longrightarrow \mathbb{R}$ the function θ_Z lifted to M. Then

$$dd^c\varphi(z,b) = \theta\cdot\omega_B + dd^c\theta\cdot|b|^2 - \sqrt{-1}\partial\theta\wedge\sum_i x_i d\overline{x}_i + \sqrt{-1}\overline{\partial}\theta\wedge\sum_i \overline{x}_i dx_i, \ (4.1)$$

where x_i are standard coordinates on B. For $|x_i|$ sufficiently small comparing with C_1, the last two terms are small comparing with $\theta\omega_B$. Similarly, when $|b|^2$ is small comparing with C_2, the sum $\theta\omega_B + dd^c\theta|b|^2$ obviously satisfies the inequalities (2.2), hence the form

$$(\theta\omega_B + dd^c\theta|b|^2) \wedge \omega^{q-1}$$

is positive. For such b, the function φ is ω^q-plurisubharmonic. We proved Sublemma 4.4. ∎

Now Lemma 4.3 is finished trivially. Consider a locally finite covering W_i of Z, such that some neighbourhood of Z is covered by a union of open sets $M_i \cong W_i \times B_i$, where B_i are isomorphic to open balls in $\mathbb{C}^n$. Taking a function $\theta_i : \ W_i \longrightarrow \mathbb{R}$ vanishing on a boundary of W_i with all its derivatives and satisfying the conditions of Sublemma 4.4, we obtain a strictly ω^q-plurisubharmonic function on M_i which vanishes on $\partial W_i \times B_i$, with all its derivatives. Summing up these functions, we obtain a function φ which is strictly ω^q-plurisubharmonic on some neighbourhood of Z_0 and vanishes on the boundary of M, with all its derivatives. Lemma 4.3 is proven. ∎

Proposition 4.5: Let $Z \subset M$ be a closed complex subvariety of a Kähler manifold. Then there exists a strictly ω^q-plurisubharmonic function on an open neighbourhood U of Z in M.

Proof: For $\dim Z = 0$, the distance function $x \longrightarrow \mathrm{dist}(x,Z)$ is strictly plurisubharmonic, and consequently ω^q-plurisubharmonic in some neighbourhood of Z. Using induction by $\dim Z$, we may assume that Proposition 4.5

is proven for $Z_1 = \mathrm{Sing}(Z)$ (the singular set of Z). Let $U_1 \supset Z_1$ be an open neighbourhood of Z_1, and $\varphi_1 : \ U_1 \longrightarrow \mathbb{R}$ a strictly ω^q-plurisubharmonic function. Consider a smaller neighbourhood $V_1 \subset U_1$, such that its closure $\overline{V}_1$ is compact and contained in U_1.[1] Let $\xi : \ M \mapsto [0,1]$ be a smooth function which is equal 1 on V_1 and 0 on $M \backslash U_1$. Consider a neighbourhood $W_1 \Subset V_1$ of Z_1 with smooth boundary, and let φ_0 be a strictly ω^q-plurisubharmonic function defined in a neighbourhood of $Z \backslash W_1$, and vanishing on ∂W_1 with all derivatives (Lemma 4.3). We extend φ_0 over W_1 by setting $\varphi_0\big|_{W_1} = 0$. Then, φ_0 and $\xi \varphi_1$ are defined in an open neighbourhood U of Z.
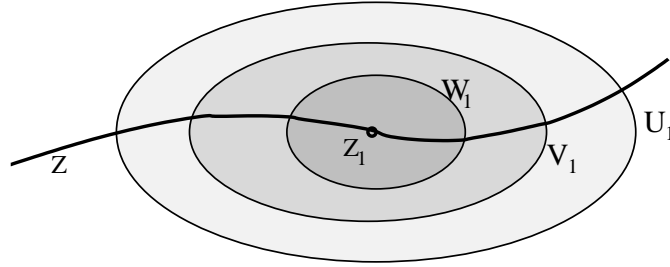
- V_1: here $\xi = 1$, φ_0 is ω^q-plurisubharmonic, and φ_1 strictly ω^q-plurisubharmonic.

- $U \backslash U_1$: here $\xi = 0$, and φ_0 is strictly ω^q-plurisubharmonic

- $U_1 \backslash V_1$: here φ_0 is strictly ω^q-plurisubharmonic, hence $C\varphi_0 + \xi\varphi_1$ is stricty ω^q-plurisubharmonic, for $C \gg 0$.

Figure 1: Gluing ω^q-plurisubharmonic functions around the singular set of Z

For a sufficiently big constant $C > 0$, the sum $\varphi = C\varphi_0 + \xi\varphi_1$ is strictly ω^q-plurisubharmonic. Indeed, on V_1, $\varphi = C\varphi_0 + \xi\varphi_1 = C\varphi_0 + \varphi_1$ is ω^q-plurisubharmonic, because both of its summands are. Outside of U_1, $\varphi = \varphi_o$, and this is also ω^q-plurisubharmonic. It remains to show that φ is ω^q-plurisubharmonic on the compact set $X := \overline{U_1 \backslash V_1}$, for C sufficiently big. However, φ_0 is strictly ω^q-plurisubharmonic on this set, hence satisfies

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[1]This property is usually denoted as $V_1 \Subset U_1$.

$dd^c\varphi_0 \wedge \omega^{q-1} > \varepsilon\omega^q$. Therefore, the sum $C\varphi_0 + \xi\varphi_1$ is ω^q-plurisubharmonic for

$$C > \varepsilon^{-1}\|\partial\overline{\partial}(\xi\varphi_1)\|.$$

Here $\|\partial\overline{\partial}(\xi\varphi_1)\|(x) = -\min\{\alpha_i\}$, where $\{\alpha_i\}$ are eigenvalues of the pseudo-Hermitian form $\partial\overline{\partial}\xi\varphi_1$ at $x \in X$. We proved Proposition 4.5. ∎

4.3 Almost plurisubharmonic functions

Recall that an L^1-integrable function $\chi : M \longrightarrow \mathbb{R}$ on a complex manifold is called **almost plurisubharmonic** if the current $dd^c\chi + C\omega$ is positive, for some $C > 0$. In [D1], Demailly proved the following important theorem.

Theorem 4.6: Let $Z \subset M$ be a compact complex subvariety of a complex manifold M. Then there exists an almost plurisubharmonic function

$$\chi : M \longrightarrow [-\infty, C[$$

which is smooth outside of Z and satisfies $\lim\limits_{x \longrightarrow Z} \chi(z) = -\infty$ ∎

Using Theorem 4.6, we can finish the proof of Theorem 4.1. This argument is lifted from [D2], almost without changes. Let $\varphi : V \longrightarrow \mathbb{R}$ be a strictly ω^q-plurisubharmonic function in a neighbourhood of $Z \subset M$. Such a function exists by Proposition 4.5. Rescaling and replacing V with a smaller neighbourhood if necessary, we may assume that $dd^c\varphi \wedge \omega^{q-1} > \omega^q$. Using Theorem 4.6, we construct an almost plurisubharmonic function χ on M, with a pole in Z. Since χ is almost plurisubharmonic, $dd^c\chi + C\omega$ is positive. Then

$$(Cdd^c\varphi + dd^c\chi) \wedge \omega^{q-1} > C\omega^q + dd^c\chi \wedge \omega^{q-1} = (C\omega + dd^c\chi) \wedge \omega^{q-1}.$$

The last form is positive, because $dd^c\chi + C\omega$ is positive. Therefore, $\chi_1 := C\varphi + \chi$ is ω^q-plurisubharmonic. Since

$$\lim_{x \longrightarrow Z} \chi(x) = -\infty,$$

and χ is continuous outside of Z, for sufficiently small set A, the preimage $W := \chi_1^{-1}([-\infty, A])$ contains an open neighbourhood of Z. Assume that $\varphi < B - \varepsilon$ on W. Then

$$\psi := \max_{\frac{\varepsilon}{3}}(\varphi - B, \chi_1 - A)$$

is equal to $\chi_1 + A$ in a neighbourhood of ∂W, and to $\varphi - B$ in a neighbourhood of Z. Here $\max_{\frac{\varepsilon}{3}}$ denotes the regularized maximum (see the proof of Lemma 3.8). By Corollary 3.13, ψ is strictly ω^q-plurisubharmonic. The function ψ is smooth on outside of Z, being a regularized maximum of two smooth functions, and it is smooth in a neighbourhood of Z, because $\psi = \varphi - B$ in a small neighbourhood of Z. Also, ψ is non-positive on W, and equal 0 on ∂W, hence it is proper and exhausting on W. We proved Theorem 4.6. ∎

We also proved the following result.

Corollary 4.7: Let M be a Kähler manifold, and $Z \subset M$ a complex subvariety, $\dim_{\mathbb{C}} Z \leqslant q$. Then there exists an open neighbourhood V of Z and a strictly ω^q-plurisubharmonic function $\chi_1 : V \longrightarrow [-\infty, 0]$ which is smooth outside of Z, and satisfies

$$\lim_{x \longrightarrow Z} \varphi(z) = -\infty$$

Proof: The function $\chi_1 := C\varphi + \chi$ defined above obviously satisfies these properties. ∎

5   A proof of Sibony's lemma on L^1-integrability of positive currents

As an application of Theorem 4.6, we give a proof of the following classical result, a more general version of which is due to Sibony ([Sib]).

Theorem 5.1: (Sibony's lemma). Let M be a Kähler manifold, $\dim_{\mathbb{C}} M = n$, and $Z \subset M$ a compact complex subvariety of dimension at most $p - 1$. Consider a weakly positive, closed, locally L^1-integrable $(n - p, n - p)$-form θ on $M \backslash Z$. Then θ is locally integrable on M.

Remark 5.2: The usual proof of this lemma does not require M to be Kähler, and Z to be compact. Instead, one introduces coordinates and uses the slicing method. In other geometric situations (on hyperkähler manifolds, for example) there are no flat coordinates, and slicing fails. Indeed, a typical hyperkähler manifold does not have any non-trivial hyperkähler subvarieties, even locally. The arguments used below for a coordinate-free

proof of Sibony's lemma can be employed in hyperkähler geometry, to obtain results on stability of derived direct images of coherent sheaves (see [V1], [V3]).

To prove Theorem 5.1, we use the following proposition.

Proposition 5.3: Let M be a Kähler manifold, and $Z \subset M$ a complex subvariety, $\dim_{\mathbb{C}} Z < p$. Then there exists an open neighbourhood U of Z, and a sequence $\{\eta_i\}$ of strongly positive, exact, smooth (p,p)-forms on U satisfying the following.

(i) For any open subset $V \subset U$, with the closure $\overline{V}$ compact and not intersecting Z, the restriction $\eta_i\big|_V$ stabilizes as $i \longrightarrow \infty$. Moreover, $\eta_i\big|_V$ is strictly positive for $i \gg 0$ (that is, lies in the inner part of the strongly positive cone; see Subsection 1.2 for details).

(ii) For all i, $\eta_i = 0$ in some neighbourhood of Z.

(iii) The limit $\eta = \lim \eta_i$ is a strictly positive current on U.

(iv) The forms η_i can be written as $\eta_i = dd^c \varphi_i \wedge \omega^{p-1}$, where φ_i are smooth functions on U. On any compact set not intersecting Z, the sequence $\{\varphi_i\}$ stabilizes as $i \longrightarrow \infty$.

Proof: By Corollary 4.7, there exists a strictly ω^q-plurisubharmonic function φ on a neighbourhood U of Z, smooth outside of Z, and satisfying

$$\lim_{x \longrightarrow Z} \varphi(z) = -\infty.$$

Using the regularized maximum, we obtain a sequence of smooth ω^q-plurisubharmonic functions $\varphi_N := \max_{\varepsilon}(-N, \varphi)$. Let $\eta_i := dd^c \varphi_i \wedge \omega^{p-1}$. This form is positive, because φ_i is ω^q-plurisubharmonic, and equal to φ on the set

$$\{x \in U \ \mid \ \varphi(x) > -i + \varepsilon\}.$$

This gives the condition (i) of Proposition 5.3. The condition (ii) is also clear, because $dd^c \varphi_i = 0$ on the set

$$\{x \in U \ \mid \ \varphi(x) < -i - \varepsilon\}.$$

The conditions (iii) and (iv) are apparent from the construction. We proved Proposition 5.3. ∎

To finish the proof of Theorem 5.1, we choose a neighbourhood $U \supset Z$, with compact closure and a smooth boundary ∂U, and admitting a sequence $\{\eta_i\}$ of smooth forms satisfying Proposition 5.3. Denote by η the current $\lim_{i \longrightarrow \infty} \eta_i$. Since η is strictly positive, and θ is positive, to prove that θ is locally integrable, it suffices to establish the inequality

$$\int_K \theta \wedge \eta < C \qquad (5.1)$$

for any compact set $K \subset U$ not intersecting Z, and a constant C independent from the choice of K. However,

$$\int_K \theta \wedge \eta = \lim_{i \longrightarrow \infty} \int_K \theta \wedge \eta_i,$$

hence (5.1) would follow from a similar universal bound $\int_K \theta \wedge \eta_i < C$. The integral $\int_U \theta \wedge \eta_i < C$ is well defined, because η_i vanish in a neighbourhood of Z, and satisfies

$$\int_K \theta \wedge \eta_i \leqslant \int_U \theta \wedge \eta_i$$

for any $K \subset U$. Therefore, to prove (5.1) it suffices to show that there exists a universal bound

$$\int_U \theta \wedge \eta_i < C, \qquad (5.2)$$

with C independent from the choice of i.

However,

$$\int_U \theta \wedge \eta_i = \int_{\partial U} \theta \wedge d^c \varphi_i \wedge \omega^{p-1}. \qquad (5.3)$$

Since $\{\varphi_i\}$ stabilizes in a neighbourhood of ∂U, as follows from Proposition 5.3 (iv), the integral (5.3) also stabilizes. Therefore, it is universally bounded. This proves (5.2). We have shown that θ is integrable. ∎

References

[Ba]   D. Barlet, *Convexité de l'espace des cycles*, Bull. Soc. Math. France, **106** (1978), 373-397.

[BGV]   Berline, N., Getzler, E., Vergne, M., *Heat kernels and Dirac operators*, Grundlehren Text Editions. Springer-Verlag, Berlin, 1992.

[Bl]   Z. Błocki, *Weak solutions to the complex Hessian equation*, Ann. Inst. Fourier 55 (2005), 1735-1756

[C]   M. Colţoiu, *Complete locally pluripolar sets,* J. reine angew. Math., 412 (1990), 108-112.

[D1]   J.-P. Demailly, *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète,* Ann. Sci. Ecole Norm. Sup. 4e Sér. 15 (1982) 457–511.

[D2]   Demailly, Jean-Pierre, *Cohomology of q-convex spaces in top degrees*, Math. Z. 204 (1990), no. 2, 283–295.

[D3]   Demailly, Jean-Pierre, L^2 *vanishing theorems for positive line bundles and adjunction theory*, Lecture Notes of a CIME course on "Transcendental Methods of Algebraic Geometry" (Cetraro, Italy, July 1994), arXiv:alg-geom/9410022, and also Lecture Notes in Math., 1646, pp. 1–97, Springer, Berlin, 1996

[DF]   Diederich, Klas; Fornæss, John Erik; *Smoothing q-convex functions and vanishing theorems*, Invent. Math. 82 (1985), no. 2, 291–305.

[GW]   R. E. Greene and H. Wu, C^∞-*approximations of convex, subharmonic, and plurisubharmonic functions*, Ann. Scient. Ec. Norm. Sup. 12 (1979), 47-84.

[Ha]   Harvey, F. Reese, *Spinors and calibrations,* Perspectives in Mathematics, 9, Academic Press, Inc., Boston, MA, 1990.

[HL1]   R. Harvey, B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47-157.

[HL2]   R. Harvey, B. Lawson, *An Introduction to Potential Theory in Calibrated Geometry,* arXiv:0710.3920, 45 pages

[HL3]   R. Harvey, B. Lawson, *Duality of Positive Currents and Plurisubharmonic Functions in Calibrated Geometry,* arXiv:0710.3921, 29 pages.

[He]   Hein, Georg, *Computing Green currents via the heat kernel,* J. Reine Angew. Math. 540 (2001), 87–104.

[J]   Joyce, Dominic D., *Riemannian holonomy groups and calibrated geometry,* Oxford Graduate Texts in Mathematics, 12. Oxford University Press, Oxford, 2007

[Kh1]   Khusanov, Z. Kh., *Maximal q-subharmonic functions,* (Russian) Dokl. Akad. Nauk UzSSR 1988, no. 4, 9–10.

[Kh2]   Khusanov, Z. Kh., *Capacity properties of q-subharmonic functions. I,* (Russian) Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk 1990, no. 1, 41–45, 97.

[Kh3]   Khusanov, Z. Kh., *Capacity properties of q-subharmonic functions. II,* (Russian) Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk 1990, no. 5, 28–33, 94.

[NR]   Terence Napier; Mohan Ramachandran *The Bochner-Hartogs dichotomy for weakly 1-complete Kähler manifolds*, Annales de l'institut Fourier, 47 no. 5 (1997), p. 1345-1365

[O]   T. Ohsawa, *A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds,* Invent. Math., 63 (1981), 335-354 and 66 (1982), 391-393.

[P]   Peternell, M., *Continuous q-convex exhaustion functions*, Invent. Math. 85 (1986), no. 2, 249–262.

[Sib]   Sibony, Nessim, *Quelques problemes de prolongement de courants en analyse complexe,* Duke Math. J. 52, 157-197 (1985).

[V1]   Verbitsky, M., *Hyperholomorpic connections on coherent sheaves and stability*, 40 pages, math.AG/0107182

[V2]   Verbitsky, M., *Manifolds with parallel differential forms and Kähler identities for G_2-manifolds,* arXiv:math/0502540, 34 pages

[V3]   M. Verbitsky, *Positive forms on hyperkahler manifolds,* arXiv:0801.1899, 33 pages

[W]   H. Wu, *On certain Kähler manifolds which are q-complete,* Complex Analysis of Several Variables, Proceedings of Symposia in Pure Mathematics, 41, Amer. Math. Soc., Providence (1984), 253-276.

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